A level set method using the signed distance function (Numerical Solution of Partial Differential Equations and Related Topics)

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A level set method using the signed distance function

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1 Introduction

In this paper, we consider the following moving boundary problems. Let Ω be a bounded domain of $\mathbf{R}^m$ $(m\geq 2)$ and let $\Gamma(t)\subset\Omega$ be a moving closed hypersurface which divides Ω into two open subset $\Omega^+(t)$ and $\Omega^-(t)$ as

$$\Omega\setminus\Gamma(t)=\Omega^+(t)\cup\Omega^-(t),\qquad \Omega^+(t)\cap\Omega^-(t)=\emptyset,\qquad \overline{\Omega^-(t)}\cap\partial\Omega=\emptyset,$$

where t is a time variable. In this paper, we call $\Gamma(t)$ a moving interface and, for our convenience, we call $\Omega^+(t)$ and $\Omega^-(t)$ the outer and inner domain of $\Gamma(t)$ respectively. For $\boldsymbol{x}\in\Gamma(t)$, $\boldsymbol{\nu}(\boldsymbol{x},t)$, $\kappa(\boldsymbol{x},t)$ and $v(\boldsymbol{x},t)$ stand for the inward unit normal vector, the sum of the principal curvatures and the inward normal velocity of $\Gamma(t)$ at $\boldsymbol{x}$, respectively, where the signs of principal curvatures are nonnegative if $\Omega^-(t)$ is convex.

The following problem is the object of our study.

Problem 1.1 *Let $\Gamma^0\subset\Omega$ be a given initial interface. For given $\mu\geq 0$, $\boldsymbol{u}\in C^1(\overline{\Omega}\times[0,T];\mathbf{R}^m)$ and $g\in C^1((\overline{\Omega}\times[0,T];\mathbf{R})$ $(T>0)$, find $\Gamma(t)$ $(0\leq t\leq T)$ such that*

$$\begin{cases} v(\boldsymbol{x},t) &=& \mu\kappa(\boldsymbol{x},t)+\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\nu}(\boldsymbol{x},t)+g(\boldsymbol{x},t), \quad (\boldsymbol{x}\in\Gamma(t),\ 0<t\leq T)\\ \Gamma(0) &=& \Gamma^0. \end{cases}$$

In the case that $\boldsymbol{u}\equiv 0$ and $g\equiv 0$, Problem 1.1 is called the curve shortening problem for $m=2$ and the mean curvature flow problem for $m\geq 3$ ([1], [2], [3], [4], [5] etc.). Roughly speaking, the coefficient μ corresponds to the surface tension of $\Gamma(t)$ physically.

If $\mu=0$ and $g\equiv 0$, this problem stands for the motion of a surface in the velocity field $\boldsymbol{u}$. The term g stands for some external forces. In particular, when $\mu=0$, $\boldsymbol{u}\equiv 0$ and $g\equiv 1$, this is the equation of growth with constant speed.

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A lot of numerical methods for moving boundary problems including Problem 1.1 have been proposed. We make mention of only a few of them here. In general, a numerical method for a moving boundary problem incline to be complicated and to include some ad hoc procedures. It means not only less applicability but also difficulty in mathematical analysis of the method. It is usually not easy to show convergence of numerical interfaces to exact one.

There are only a few mathematical results on convergence of interfaces, as far as the author knows. A front tracking method using a special, but not ad hoc, finite difference scheme for the curve shortening problem is proposed in [6], [7] by the author, and its convergence is proved. This method can be applied other two dimensional moving boundary problems. But it is not easy to apply it to three dimensional problems. In multi-dimensional case, for Problem 1.1 without $\boldsymbol{u}$, a finite element method using approximation by a reaction diffusion equation is proposed by Nochetto and Verdi [11] and its convergence is also proved. Their method has not only good theoretical background but also good practicality. But it is difficult to apply their method to other moving boundary problems, because it is strongly based on the nature of the problem.

We propose a finite difference–level set method using the signed distance function for Problem 1.1. The aim of this paper is to give a convergence theorem for the case $\mu = 0$ (Theorem 4.2). The proof is based on the Lipschitz continuity of the signed distance function (Proposition 2.2) and on the discrete maximum principle (Lemma 3.2). The basic lemmas and propositions are valid for $\mu \geq 0$. Because of the restriction of pages, almost of the proofs of lemmas and propositions are omitted in this article. The complete proofs are found in [8].

We briefly mention the other sides of this method which can not be discussed in this paper. For the case $\mu > 0$, we have no convergence theorem but numerical convergence is observed in [9] and [10]. An effective algorithm to realize our discretized problem in practical numerical computation is also proposed in [9] and [10]. In addition to these reliability and practicality, we can expect wide applicability of this method because of the use of the signed distance function. For instance, the signed distance function does not depend on the problem but is defined only by the shape of the moving boundary.

The paper is organized as follows. In the next section, we introduce the signed distance function and state some its properties. A level set formulation of Problem 1.1 is also given in § 2. In § 3, its finite difference approximation is considered and its basic properties and some lemmas are given. In the last section, we give a convergence theorem for the case $\mu = 0$. This is the main result of this paper.

2 Signed distance function

For a moving interface $\Gamma(t)$ $(0 \leq t \leq T)$ as in § 1, we define the signed distance function as follows.

Definition 2.1 (signed distance function)

$$d(x,t) := \begin{cases} +\mathrm{dist}(x,\Gamma(t)), & (x \in \overline{\Omega^+(t)},\ 0 \le t \le T), \\ -\mathrm{dist}(x,\Gamma(t)), & (x \in \Omega^-(t),\ 0 \le t \le T). \end{cases}$$

We define

$$\mathcal{M} := \bigcup_{0\le t\le T} \Gamma(t) \times \{t\} \quad \subset \mathbf{R}^m \times \mathbf{R}, \tag{2.1}$$

and assume that the following conditions:

$$\begin{array}{l} \mathcal{M} \text{ is a } C^1\text{-class hypersurface of } \mathbf{R}^m \times \mathbf{R}.\ \Gamma(t) \text{ is of } C^2\text{-class and its} \\ \text{principal curvatures and principal directions are continuous on } \mathcal{M}. \end{array} \tag{2.2}$$

For $\varepsilon > 0$, the ε-neighborhood of $\Gamma(t)$ and $\mathcal{M}$ are denoted by

$$N^\varepsilon(\Gamma(t)) := \{x \in \Omega;\ \mathrm{dist}(x,\Gamma(t)) < \varepsilon\}, \qquad N^\varepsilon(\mathcal{M}) := \{(x,t);\ x \in N^\varepsilon(\Gamma(t)),\ t \in [0,T]\}.$$

There exists a positive constant ε^* such that the map: $(\boldsymbol{x},\rho) \mapsto \boldsymbol{x} - \rho\boldsymbol{\nu}(\boldsymbol{x})$ is a C^1-diffeomorphism from $\Gamma(t) \times [-\varepsilon^*,\varepsilon^*]$ to $\overline{N^{\varepsilon^*}(\Gamma(t))} \subset \Omega$ for $0 \le t \le T$. Let $\bar{\boldsymbol{x}} \in \Gamma(t)$ be the foot of the perpendicular to $\Gamma(t)$ from $\boldsymbol{x} \in N^{\varepsilon^*}(\Gamma(t))$. Under the assumptions (2.2), it is known that d, d_{x_i}, $d_{x_ix_j}$ and d_t are continuous in $\overline{N^{\varepsilon^*}(\mathcal{M})}$, and

$$\bar{\boldsymbol{x}} = \boldsymbol{x} - d(\boldsymbol{x},t)\nabla d(\boldsymbol{x},t), \quad \nabla d(\boldsymbol{x},t) = -\boldsymbol{\nu}(\bar{\boldsymbol{x}},t), \quad d_t(\boldsymbol{x},t) = v(\bar{\boldsymbol{x}},t) \qquad ((\boldsymbol{x},t) \in \overline{N^{\varepsilon^*}(\mathcal{M})}),$$

$$\kappa(\boldsymbol{y},t) = \Delta d(\boldsymbol{y},t) \qquad ((\boldsymbol{y},t) \in \mathcal{M}).$$

We give some propositions for a fixed surface Γ, where the signed distance function of Γ is defined similarly.

Proposition 2.2 *For a closed hypersurface Γ in $\mathbf{R}^m$ and its signed distance function $d(\cdot)$, the following inequality holds:*

$$|d(\boldsymbol{x}) - d(\boldsymbol{y})| \le |\boldsymbol{x} - \boldsymbol{y}| \qquad (\boldsymbol{x},\ \boldsymbol{y} \in \mathbf{R}^m).$$

For two compact subset K_1 and K_2 in $\mathbf{R}^m$, $\mathrm{d_H}(K_1,K_2)$ stands for the Hausdorff distance between K_1 and K_2:

$$\mathrm{d_H}(K_1,K_2) := \max\left\{\max_{\boldsymbol{x}\in K_1} \mathrm{dist}(\boldsymbol{x},K_2),\ \max_{\boldsymbol{y}\in K_2} \mathrm{dist}(\boldsymbol{y},K_1)\right\}.$$

We remark the following equality which is one of well-known properties of the Hausdorff distance:

$$\mathrm{d_H}(K_1,K_2) = \sup_{\boldsymbol{x}\in\mathbf{R}^m} |\mathrm{dist}(\boldsymbol{x},K_1) - \mathrm{dist}(\boldsymbol{x},K_2)| \tag{2.3}$$

For a fixed compact hypersurface Γ of C^2-class, we define $\boldsymbol{\nu}(\boldsymbol{x})$, $d(\boldsymbol{x})$, ε^* etc. as for a moving interface.

Proposition 2.3 *Let* Γ *be a compact hypersurface of* C^2*-class in* $\mathbf{R}^m$*. If a closed subset* $K \subset \overline{N^{\varepsilon^*}(\Gamma)}$ *satisfies* $\Gamma = \{\boldsymbol{x} - d(\boldsymbol{x})\nabla d(\boldsymbol{x});\ \boldsymbol{x} \in K\}$*, then* $\mathrm{d_H}(K, \Gamma) = \max_{\boldsymbol{x}\in K} |d(\boldsymbol{x})|$.

Under the condition (2.2), we define $C_v := \max_{(\boldsymbol{x},t)\in\mathcal{M}} |v(\boldsymbol{x},t)|$, and give the following proposition:

Proposition 2.4

$$|d(\boldsymbol{x},t_1) - d(\boldsymbol{x},t_2)| \leq C_v(t_2 - t_1) \qquad (\boldsymbol{x} \in \overline{\Omega},\ 0 \leq t_1 \leq t_2 \leq T).$$

Let $\Gamma(t)$ $(0 \leq t \leq T)$ be a sufficiently smooth solution of Problem 1.1 and let $d(\boldsymbol{x},t)$ be the signed distance function of $\Gamma(t)$. They satisfy the following equalities:

$$\Gamma(t) = \{\boldsymbol{x} \in \Omega;\ d(\boldsymbol{x},t) = 0\}, \qquad (0 \leq t \leq T),$$

$$\left\{\begin{array}{rcll} d_t(\boldsymbol{x},t) &=& \mu\Delta d(\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla d(\boldsymbol{x},t) + g(\boldsymbol{x},t) & (d(\boldsymbol{x},t) = 0,\ \boldsymbol{x} \in \Omega,\ 0 < t \leq T), \\ d(\boldsymbol{x},t) &=& \pm\min\{|\boldsymbol{x} - \boldsymbol{y}|;\ d(y,t) = 0\} & (\pm d(\boldsymbol{x},t) > 0,\ \boldsymbol{x} \in \Omega,\ 0 < t \leq T), \\ d(\boldsymbol{x},0) &=& d^0(\boldsymbol{x}) & (\boldsymbol{x} \in \Omega), \end{array}\right. \tag{2.4}$$

where d^0 is the signed distance function of Γ^0. For small $\varepsilon > 0$, (2.4) is formally approximated by the following problem:

$$\left\{\begin{array}{rcll} d_t^\varepsilon(\boldsymbol{x},t) &=& \mu\Delta d^\varepsilon(\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla d^\varepsilon(\boldsymbol{x},t) + g(\boldsymbol{x},t) & (|d^\varepsilon(\boldsymbol{x},t)| < \varepsilon,\ \boldsymbol{x} \in \Omega,\ 0 < t \leq T), \\ d^\varepsilon(\boldsymbol{x},t) &=& \pm\min\{|\boldsymbol{x} - \boldsymbol{y}|;\ d^\varepsilon(\boldsymbol{y},t) = 0\} & (\pm d^\varepsilon(\boldsymbol{x},t) \geq \varepsilon,\ \boldsymbol{x} \in \Omega,\ 0 < t \leq T), \\ d^\varepsilon(\boldsymbol{x},0) &=& d^0(\boldsymbol{x}) & (\boldsymbol{x} \in \Omega), \end{array}\right. \tag{2.5}$$

We expect that $\Gamma^\varepsilon(t) := \{\boldsymbol{x} \in \Omega;\ d^\varepsilon(\boldsymbol{x},t) = 0\}$ approximates $\Gamma(t)$ as ε tends to zero. Although any mathematical result for the problem (2.5) has not been obtained yet, in the next section, we proceed to the next step: discretization of (2.5).

3 Finite difference approximation

In this section, we consider a finite difference approximation of the problem (2.5). Although we consider two dimensional problem in the following sections for simplicity, all arguments are valid in the three dimensional case.

We consider a uniformly divided finite difference mesh on the $\boldsymbol{x} = (x_1, x_2)$-plane. We assume the following assumptions just for simplicity again, but they are not essential. We assume that $\Omega = (0,1) \times (0,1)$, and that x_1 and x_2 directions are both divided by a same mesh size $h = 1/n$ $(n \in \mathbf{N})$. We define $\Omega_h := (h, 1-h) \times (h, 1-h)$.

The nodal points and sets of nodal points are denoted by $\boldsymbol{\xi}_{ij} := (ih, jh)$ and

$$\omega_h := \{\boldsymbol{\xi}_{ij};\ i = 0,1,\cdots,n,\ j = 0,1,\cdots,n\}, \quad \overset{\circ}{\omega}_h := \{\boldsymbol{\xi}_{ij};\ i = 1,2,\cdots,n-1,\ j = 1,2,\cdots,n-1\}.$$

For each closed square $[ih,(i+1)h]\times[jh,(j+1)h]$ contained in $\overline{\Omega}$, we divide it into two right triangles of three sides $(h,h,\sqrt{2}h)$. (There are two way to divide a square into two right triangle. We fix one of them for each square.) Let $\mathcal{T}_h=\{e\}$ be the set of all such closed right triangles, and, for a set $K\subset\overline{\Omega}$, we define

$$S_h(K):=\{\boldsymbol{p}\in\omega_h;\ \boldsymbol{p}\in^{\exists}e\in\mathcal{T}_h,\ e\cap K\neq\emptyset\},$$

in particular, we define $S_h^2(K):=S_h(S_h(K))$ and $S_h(\boldsymbol{x}):=S_h(\{\boldsymbol{x}\})$ for $\boldsymbol{x}\in\overline{\Omega}$. For $\mathcal{P}\subset\omega_h$, we also define

$$T_h(\mathcal{P}):=\{\boldsymbol{x}\in\overline{\Omega};\ \boldsymbol{x}\in^{\exists}e\in\mathcal{T}_h,\ (e\cap\omega_h)\subset\mathcal{P}\}.$$

We define the following spaces of piecewise linear functions on $\mathcal{T}_h$:

$$V_h:=\{w_h\in C^0(\overline{\Omega});\ w_h \text{ is linear function on each } e\in\mathcal{T}_h.\},\qquad \overset{\circ}{V}_h:=\{w_h|_{\overline{\Omega_h}};\ w_h\in V_h\}.$$

The function space V_h is the same one which is known as the $P1$-element space in the finite element methods. An interpolation operator from $C^0(\overline{\Omega})$ to V_h is defined as

$$\Pi_h w(\boldsymbol{p}):=w(\boldsymbol{p})\qquad (w\in C^0(\overline{\Omega}),\ \boldsymbol{p}\in\omega_h).$$

We fix a time increment $\Delta t>0$. For brief description, we write

$$\Gamma^k:=\Gamma(k\Delta t),\qquad \boldsymbol{\nu}^k(\boldsymbol{x}):=\boldsymbol{\nu}(\boldsymbol{x},k\Delta t),\qquad d^k(\boldsymbol{x}):=d(\boldsymbol{x},k\Delta t),$$

$$\boldsymbol{u}^k(\boldsymbol{x})={}^t(u_1^k(\boldsymbol{x}),u_2^k(\boldsymbol{x})):=\boldsymbol{u}(\boldsymbol{x},k\Delta t),\qquad g^k(\boldsymbol{x}):=g(\boldsymbol{x},k\Delta t),$$

for $k=0,1,2,\cdots$. We consider $d_h^k\in V_h$ which is a finite difference approximation of d^k for $k=0,1,\cdots,[T/\Delta t]$. Approximations of Γ^k and $\Omega^{\pm}(t)$ are defined by

$$\Gamma_h^k:=\{\boldsymbol{x}\in\Omega;\ d_h^k(\boldsymbol{x})=0\},\qquad \Omega_h^{k,\pm}:=\{\boldsymbol{x}\in\Omega;\ \pm d_h^k(\boldsymbol{x})>0\},$$

where we remark that Γ_h^k is a linear segment on each triangle. For $\varepsilon>0$, we define a discretized ε-neighborhood ω_0^k of Γ_h^k as follows:

$$\omega_\pm^k:=\{\boldsymbol{p}\in\omega_h;\ \pm d_h^k(\boldsymbol{q})\geq\varepsilon\ (^\forall\boldsymbol{q}\in S_h^2(\boldsymbol{p}))\},\qquad \omega_0^k:=\omega_h\setminus(\omega_+^k\cup\omega_-^k).$$

We remark that ω_0^k has the following characterization:

$$\omega_0^k=S_h^2(N_h^\varepsilon(k)\cup\Gamma^k),\tag{3.1}$$

where $N_h^\varepsilon(k):=\{\boldsymbol{p}\in\omega_h;\ |d_h^k(\boldsymbol{p})|<\varepsilon\}$ (see [9] and [10]).

Using the standard five points finite difference formula, a discrete Laplacian $\triangle_h$ as a linear operator from V_h to $\overset{\circ}{V}_h$ is defined as:

$$(\triangle_h w_h)(\boldsymbol{\xi}_{ij}):=\frac{1}{h^2}(w_h(\boldsymbol{\xi}_{i+1,j})+w_h(\boldsymbol{\xi}_{i-1,j})+w_h(\boldsymbol{\xi}_{i,j+1})+w_h(\boldsymbol{\xi}_{i,j-1})-4w_h(\boldsymbol{\xi}_{ij}))$$
$$(w_h\in V_h,\ i,\ j=1,2,\cdots,n-1).$$

For a given vector field $\boldsymbol{u}(\boldsymbol{x},t)$ and a time increment $\Delta t > 0$, we also define an up-wind gradient operator from V_h to $(\overset{\circ}{V}_h)^2$ as follows. For $w_h \in V_h$,

$$(\delta^k_{h,l} w_h)(\boldsymbol{\xi}_{ij}) := \begin{cases} \dfrac{1}{h}(w_h(\boldsymbol{\xi}_{ij} + h\boldsymbol{e}_l) - w_h(\boldsymbol{\xi}_{ij})) & (u^k_l(\boldsymbol{\xi}_{ij}) \le 0) \\ \dfrac{1}{h}(w_h(\boldsymbol{\xi}_{ij}) - w_h(\boldsymbol{\xi}_{ij} - h\boldsymbol{e}_l)) & (u^k_l(\boldsymbol{\xi}_{ij}) > 0) \end{cases} \quad \begin{pmatrix} l = 1,2, \\ i,\ j = 1,2,\cdots,n-1 \end{pmatrix},$$

$$(\nabla^k_h w_h)(\boldsymbol{\xi}_{ij}) := {}^t\!\Big((\delta^k_{h,1} w_h)(\boldsymbol{\xi}_{ij}), (\delta^k_{h,2} w_h)(\boldsymbol{\xi}_{ij})\Big) \qquad (i,\ j = 1,2,\cdots,n-1),$$

where $\boldsymbol{e}_1 := {}^t(1,0)$ and $\boldsymbol{e}_2 := {}^t(0,1)$.

As a numerical scheme for Problem 1.1, we propose the following fully discretized problem.

Problem 3.1 *Fix parameters $h > 0$, $\Delta t > 0$ and $\varepsilon > 0$. For given $\mu \ge 0$, $\boldsymbol{u} \in C^0(\overline{\Omega} \times [0,T])^2$ and $g \in C^0(\overline{\Omega} \times [0,T])$, find $d^k_h \in V_h$ $(k = 0,1,\cdots,[T/\Delta t])$ and $\tilde{d}^k_h \in \overset{\circ}{V}_h$ $(k = 1,2,\cdots,[T/\Delta t])$ which satisfy the following equations:*

$$\begin{cases} \dfrac{\tilde{d}^{k+1}_h(\boldsymbol{p}) - d^k_h(\boldsymbol{p})}{\Delta t} = \mu \Delta_h d^k_h(\boldsymbol{p}) - \boldsymbol{u}^k(\boldsymbol{p}) \cdot \nabla^k_h d^k_h(\boldsymbol{p}) + g^k(\boldsymbol{p}) & \left(\boldsymbol{p} \in \overset{\circ}{\omega}_h,\ 0 \le k \le \dfrac{T}{\Delta t}\right), \\ d^{k+1}_h(\boldsymbol{p}) = \begin{cases} \tilde{d}^{k+1}_h(\boldsymbol{p}) & \left(\boldsymbol{p} \in \omega^k_0,\ 0 \le k \le \dfrac{T}{\Delta t}\right), \\ \pm \min\left\{|\boldsymbol{p} - \boldsymbol{y}|;\ \tilde{d}^{k+1}_h(\boldsymbol{y}) = 0,\ \boldsymbol{y} \in \overline{\Omega_h}\right\} & \left(\boldsymbol{p} \in \omega^k_\pm,\ 0 \le k \le \dfrac{T}{\Delta t}\right), \end{cases} \\ d^0_h = \Pi_h d^0, \end{cases} \tag{3.2}$$

It is obvious that this problem is explicitly solvable. More practical numerical algorithm for this problem can be found in [9] and [10].

We define the following constants and give a simple but important lemma.

$$C_u := \max_{\boldsymbol{x} \in \overline{\Omega}, 0 \le t \le T} (|u_1(\boldsymbol{x},t)| + |u_2(\boldsymbol{x},t)|), \qquad C_g := \max_{\boldsymbol{x} \in \overline{\Omega}, 0 \le t \le T} |g(\boldsymbol{x},t)|.$$

Lemma 3.2 *We suppose that $\Delta t > 0$, $h > 0$ and $\varepsilon > 0$ satisfy the following inequalities:*

$$(4\mu + C_u h)\Delta t \le h^2, \qquad C_g \Delta t < \varepsilon. \tag{3.3}$$

For $\boldsymbol{p} \in \overset{\circ}{\omega}_h$, if

$$\pm\, d^k_h(\boldsymbol{q}) \ge \varepsilon \qquad (\boldsymbol{q} \in \omega_h,\ |\boldsymbol{p} - \boldsymbol{q}| \le h), \tag{3.4}$$

then we have $\pm \tilde{d}^{k+1}_h(\boldsymbol{p}) > 0$.

Proof. This is just an application of the discrete maximum principle as follows:

$$\begin{aligned}\pm\tilde{d}_h^{k+1}(\boldsymbol{p}) &= \pm\left[d_h^k(\boldsymbol{p}) + \Delta t\mu\Delta_h d_h^k(\boldsymbol{p}) - \Delta t\boldsymbol{u}^k(\boldsymbol{p})\cdot\nabla_h^k d_h^k(\boldsymbol{p}) + \Delta t g^k(\boldsymbol{p})\right]\\ &\geq \min\{\pm d_h^k(\boldsymbol{q});\ \boldsymbol{q}\in\omega_h,\ |\boldsymbol{p}-\boldsymbol{q}|\leq h\} - \Delta t|g^k(\boldsymbol{p})| \geq \varepsilon - \Delta t C_g > 0.\end{aligned}$$

□

Since $\boldsymbol{p}\in S_h(\omega_+^k)$ or $\boldsymbol{p}\in S_h(\omega_-^k)$ satisfies (3.4), we have the following corollary.

Corollary 3.3 *Under the assumption (3.3), for all $k = 0, 1, \cdots, [T/\Delta t] - 1$, we have*

$$\{\boldsymbol{x}\in\Omega;\ \tilde{d}_h^{k+1}(\boldsymbol{x}) = 0\} = \Gamma_h^{k+1}\subset T_h(\omega_0^k), \qquad T_h(S_h(\omega_\pm^k))\subset\Omega_h^{k+1,\pm}.$$

Let $\bar{h}$ and $\underline{h}$ denote the maximum diameter of the triangular elements and the maximum radius of the circumscribed circles of the triangular elements. In our case, $\bar{h} = \sqrt{2}h$ and $\underline{h} = h/\sqrt{2}$. We define

$$e_h^k := \Pi_h d^k - d_h^k,$$

and $\tilde{\varepsilon} := \max\{\varepsilon, \bar{h}\}$, then we have the following lemmas. For their proofs, see [8].

Lemma 3.4 *We fix k ($k = 0, 1, \cdots, [T/\Delta t]$). For $\boldsymbol{p}\in\omega_0^k$, we have*

$$|d^k(\boldsymbol{p})| \leq \|e_h^k\|_\infty + 2\bar{h} + \tilde{\varepsilon}. \tag{3.5}$$

Lemma 3.5 *We suppose (2.2) and (3.3). We assume that the following inequality holds for some k,*

$$\|e_h^k\|_\infty + \underline{h} + 2\bar{h} + \tilde{\varepsilon} + C_v\Delta t \leq \varepsilon^*, \tag{3.6}$$

then we have

$$T_h(\omega_0^k)\subset\overline{\mathcal{N}_k}, \qquad \mathrm{d_H}(\Gamma^{k+1}, \Gamma_h^{k+1}) = \max_{\boldsymbol{x}\in\Gamma_h^{k+1}} |d^{k+1}(\boldsymbol{x})|,$$

where

$$\mathcal{N}_k := \bigcap_{k\Delta t\leq t\leq(k+1)\Delta t} N^{\varepsilon^*}(\Gamma(t)).$$

Lemma 3.6 *We assume (2.2), (3.3) and (3.6), then we have the following estimate:*

$$|e_h^{k+1}(\boldsymbol{p})| \leq \mathrm{d_H}(\Gamma^{k+1}, \Gamma_h^{k+1}) \qquad (\boldsymbol{p}\in\omega_+^k\cup\omega_-^k).$$

4 Convergence theorem

In this section, we assume the following conditions:

$$\mu = 0, \qquad C_u \Delta t \le h < h_0, \qquad C_g \Delta t < \varepsilon, \tag{4.1}$$

where

$$h_0 := \inf\{|\boldsymbol{x} - \boldsymbol{y}|;\ \boldsymbol{x} \in N^{\varepsilon^*}(\Gamma(t)),\ \boldsymbol{y} \in \partial\Omega,\ 0 \le t \le T\}.$$

We have the following lemma.

Lemma 4.1 *We assume the condition (4.1). Let $\Gamma(t)$ $(0 \le t \le T)$ be a solution of Problem 1.1 and we define $\mathcal{M}$ by (2.1). If $\mathcal{M}$ is a C^2-class hypersurface of $\mathbf{R}^m \times \mathbf{R}$, then there exist positive constants M_1, M_2, M_3 which are independent of Δt, h, k and $\boldsymbol{p}$ such that*

$$|e_h^{k+1}(\boldsymbol{p})| \le (1+\Delta t M_2)\|e_h^k\|_\infty + \Delta t^2 M_1 + \Delta t \tilde{\varepsilon} M_2 + \Delta t h (2\sqrt{2} M_2 + M_3) \qquad (\boldsymbol{p} \in \omega_0^k \cap \overline{\mathcal{N}}_k). \tag{4.2}$$

Proof. For $k = 0, 1, \cdots, [T/\Delta t] - 1$, we define

$$\begin{aligned} M_1^k(\boldsymbol{x}) &:= \frac{1}{\Delta t^2}\left(d^{k+1}(\boldsymbol{x}) - d^k(\boldsymbol{x}) - \Delta t d_t(\boldsymbol{x}, k\Delta t)\right), & (\boldsymbol{x} \in \overline{\mathcal{N}}_k) \\ M_2^k(\boldsymbol{x}) &:= \frac{1}{d^k(\boldsymbol{x})}\left\{-(\boldsymbol{u}^k(\bar{\boldsymbol{x}}) - \boldsymbol{u}^k(\boldsymbol{x})) \cdot \nabla d^k(\boldsymbol{x}) + (g^k(\bar{\boldsymbol{x}}) - g^k(\boldsymbol{x}))\right\}, & (\boldsymbol{x} \in \overline{\mathcal{N}}_k \setminus \Gamma^k) \end{aligned}$$

where $\bar{\boldsymbol{x}} := \boldsymbol{x} - d^k(\boldsymbol{x})\nabla d^k(\boldsymbol{x})$. By the Taylor expansion, we have

$$M_1^k(\boldsymbol{x}) = \frac{1}{2} d_{tt}(\boldsymbol{x}, (k + \theta(\boldsymbol{x}))\Delta t),$$

for some $\theta(\boldsymbol{x}) \in (0,1)$ because $d \in C^2(\overline{N^{\varepsilon^*}(\mathcal{M})})$. The value of $M_2^k(\boldsymbol{x})$ for $\boldsymbol{x} \in \Gamma^k$ is not defined, but it is possible to define it as $M_2^k \in C^0(\overline{\mathcal{N}_k})$. Hence, there exist constants M_1 and M_2 which are independent of Δt, k and $\boldsymbol{x}$ such that $|M_l^k(\boldsymbol{x})| \le M_l$ $(l = 1, 2)$ for $\boldsymbol{x} \in \overline{\mathcal{N}_k}$, $k = 0, 1, \cdots, [T/\Delta t] - 1$. Using these functions, for $\boldsymbol{x} \in \overline{\mathcal{N}_k}$, we have

$$\begin{aligned} d^{k+1}(\boldsymbol{x}) &= d^k(\boldsymbol{x}) + \Delta t d_t(\boldsymbol{x}, k\Delta t) + \Delta t^2 M_1^k(\boldsymbol{x}) \\ &= d^k(\boldsymbol{x}) + \Delta t v(\bar{\boldsymbol{x}}, k\Delta t) + \Delta t^2 M_1^k(\boldsymbol{x}) \\ &= d^k(\boldsymbol{x}) - \Delta t \boldsymbol{u}^k(\bar{\boldsymbol{x}}) \cdot \nabla d^k(\boldsymbol{x}) + \Delta t g^k(\bar{\boldsymbol{x}}) + \Delta t^2 M_1^k(\boldsymbol{x}) \\ &= d^k(\boldsymbol{x}) - \Delta t \boldsymbol{u}^k(\boldsymbol{x}) \cdot \nabla d^k(\boldsymbol{x}) + \Delta t g^k(\boldsymbol{x}) + \Delta t^2 M_1^k(\boldsymbol{x}) + \Delta t d^k(\boldsymbol{x}) M_2^k(\boldsymbol{x}). \end{aligned}$$

Furthermore, for $\boldsymbol{p} \in \mathring{\omega}_h \cap \overline{\mathcal{N}_k}$, we have

$$d^{k+1}(\boldsymbol{p}) = d^k(\boldsymbol{p}) - \Delta t \boldsymbol{u}^k(\boldsymbol{p}) \cdot \nabla_h^k \Pi_h d^k(\boldsymbol{p}) + \Delta t g^k(\boldsymbol{p}) + \Delta t^2 M_1^k(\boldsymbol{p}) + \Delta t d^k(\boldsymbol{p}) M_2^k(\boldsymbol{p}) + \Delta t h M_3^k(\boldsymbol{p}),$$

where

$$M_3^k(\boldsymbol{p}) := \frac{1}{h} \boldsymbol{u}^k(\boldsymbol{p}) \cdot (\nabla_h^k \Pi_h d^k(\boldsymbol{p}) - \nabla d^k(\boldsymbol{p})).$$

There exists a constant M_3 which is independent of Δt, h, k and $\boldsymbol{p}$ such that

$$|M_3^k(\boldsymbol{p})| \leq M_3 \qquad (\boldsymbol{p} \in \mathring{\omega}_h \cap \overline{\mathcal{N}_k},\ k = 0, 1, \cdots, [T/\Delta t] - 1).$$

For $\boldsymbol{p} \in \omega_0^k \cap \overline{\mathcal{N}}_k$, we obtain

$$\begin{aligned} e_h^{k+1}(\boldsymbol{p}) &= d^{k+1}(\boldsymbol{p}) - d_h^{k+1}(\boldsymbol{p}) \\ &= e_h^k(\boldsymbol{p}) - \Delta t \boldsymbol{u}^k(\boldsymbol{p}) \cdot \nabla_h^k e_h^k(\boldsymbol{p}) + \Delta t^2 M_1^k(\boldsymbol{p}) + \Delta t d^k(\boldsymbol{p}) M_2^k(\boldsymbol{p}) + \Delta t h M_3^k(\boldsymbol{p}). \end{aligned}$$

From this equality, Lemma 3.4 and the discrete maximum principle (similar as in Lemma 3.2), (4.2) is obtained as follows:

$$|e_h^{k+1}(\boldsymbol{p})| \leq \|e_h^k\|_\infty + \Delta t^2 M_1 + \Delta t(\|e_h^k\|_\infty + 2\sqrt{2}h + \tilde{\varepsilon}) M_2 + \Delta t h M_3.$$

□

We define $C_d := \kappa_0/(1 - \varepsilon^* \kappa_0)$, where $\kappa_0 := \max_{(\boldsymbol{x},t) \in \mathcal{M}} |\kappa(\boldsymbol{x}, t)|$. We have

$$|d(\boldsymbol{x}, t) - \Pi_h d(\boldsymbol{x}, t)| \leq C_d h^2 \qquad \left(\boldsymbol{x} \in T_h(\omega_h \cap \overline{N^{\varepsilon^*}(\Gamma(t))})\right),$$

from a well-known interpolation estimate. The following theorem is the main result of this paper.

Theorem 4.2 *Under the same assumption of Lemma 4.1, in addition, we suppose that*

$$\|e_h^0\|_\infty \to 0, \qquad \varepsilon = \varepsilon(h) \to 0, \qquad \frac{h^2}{\Delta t} \to 0 \qquad \text{as } h \to 0. \tag{4.3}$$

Then, for sufficiently small h, we have

$$\mathrm{d_H}(\Gamma^k, \Gamma_h^k) \leq e^{M_2 T} \|e_h^0\|_\infty + \frac{e^{M_2 T} - 1}{M_2} \left(\Delta t M_1 + \tilde{\varepsilon} M_2 + h(2\sqrt{2} M_2 + M_3) + C_d \frac{h^2}{\Delta t}\right) + C_d h^2. \tag{4.4}$$

In particular, suppose that there are constants $C_0 \geq 0$, $\lambda \in (0, C_u^{-1}]$ and $\sigma > C_g \lambda$ such that

$$\|e_h^0\| \leq C_0 h, \qquad \Delta t = \lambda h, \qquad \varepsilon = \sigma h,$$

then we have $\mathrm{d_H}(\Gamma^k, \Gamma_h^k) = O(h)$ as $h \to 0$.

Proof. We suppose (3.6). From Corollary 3.3 and Lemma 3.5, we have

$$\begin{aligned} \mathrm{d_H}(\Gamma^{k+1}, \Gamma_h^{k+1}) &= \max_{\boldsymbol{x} \in \Gamma_h^{k+1}} |d^{k+1}(\boldsymbol{x})| \\ &\leq \max_{\boldsymbol{x} \in \Gamma_h^{k+1}} |d^{k+1}(\boldsymbol{x}) - \Pi_h d^{k+1}(\boldsymbol{x})| + \max_{\boldsymbol{x} \in \Gamma_h^{k+1}} |e_h^{k+1}(\boldsymbol{x})| \\ &\leq C_d h^2 + \max_{\boldsymbol{p} \in \omega_0^k} |e_h^{k+1}(\boldsymbol{p})|. \end{aligned}$$

Combining this inequality and Lemma 3.6 and 4.1, provided (3.6), we obtain

$$\|e_h^{k+1}\|_\infty \leq (1+\Delta t M_2)\|e_h^k\|_\infty + \Delta t^2 M_1 + \Delta t \varepsilon M_2 + \Delta t h(2\sqrt{2}M_2 + M_3) + C_d h^2.$$

Solving this recursive inequality, we have the result (4.4), where the condition (3.6) is satisfied by each step $k = 0, 1, \cdots, [T/\Delta t] - 1$. The last assertion is clear from (4.4). □

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